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Studies of Some Curvature Operators in a Neighborhood of an Asymptotically Hyperbolic Einstein Manifold

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On an asymptotically hyperbolic Einstein manifold (M, g_0) for which the Yamabe invariant of the conformal structure on the boundary at infinity is non-negative, we show that the operators of Ricci curvature, and of Einstein curvature, are locally invertible in a neighborhood of the metric g_0. We deduce in the C^∞ case that the image of the Riemann–Christoffel curvature operator is a submanifold in a neighborhood of g_0.

Key Words: Einstein manifold; asymptotically hyperbolic; curvatures of Riemann–Christoffel Ricci, Einstein; nonlinear PDE; elliptic degenerate; asymptotic behaviour.

1. INTRODUCTION

Conformally compact manifolds form a very important class of non-compact manifolds either in Riemannian geometry or in general relativity. We are interested here in such manifolds which are asymptotically hyperbolic (their sectional curvature approches -1 at infinity) and Einstein. We show that the Ricci curvature and the Einstein curvature can be arbitrarily prescribed on the manifold in the neighborhood of an asymptotically hyperbolic Einstein metric (AHEM). We deduce that the Image of the Riemann–Christoffel operator is a submanifold near an AHEM. This result extends a previous work [D1] on the real hyperbolic space. The method to invert the Ricci or Einstein operator is an implicit function theorem in the neighborhood of the AHEM metric on some weighted Hölder spaces. The

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problem is then reduced to invert an operator of kind $\Delta+\mathscr{K}$ on a symmetric covariant two tensor where Δ is the Laplacian associated to the AHEM and $\mathscr{K}$ a term of order zero; for instance we have to invert the Lichnerowicz Laplacian in order to solve the Ricci equation. Recently J. M. Lee [L1] showed that if (M, g_0) is an AHEM with Yamabe invariant $\mathscr{Y}[\hat{g}_0]\geqslant 0$ then the first eigenvalue of the Laplacian on function is $(n-1)^2/4$. With that fact, a natural condition to invert the operator $\Delta+\mathscr{K}$ is $\inf\mathscr{K}>-(n-1)^2/4$. Note that this last condition forces the dimension n to verify $n\geqslant 10$ for the Ricci curvature and $n\geqslant 3$ for the Einstein curvature. It is of interest to study what happens in lower dimension for the Ricci curvature. In [D2], I show that the essential spectrum of the Lichnerowicz Laplacian acting on trace free symmetric covariant two tensors is the ray $[\frac{(n-1)(n-9)}{4}, +\infty[$; thus there certainly exists some obstructions to solve the Ricci equation when $n\leqslant 9$.

2. DEFINITIONS, NOTATIONS, AND CONVENTIONS

We describe here all objects we need throughout this paper.

Let $\bar{M}$ be a smooth, compact n-dimensional manifold with boundary ∂M. Let $M:=\bar{M}\setminus\partial M$ which is a noncompact manifold without boundary. We call ∂M the boundary at infinity of M. Let g be a Riemannian metric on M, we say that (M, g) is *conformally compact* if there exist a smooth defining function ρ on $\bar{M}$ (that is $\rho\in C^\infty(\bar{M})$, $\rho>0$ on M, $\rho=0$ on ∂M and $d\rho$ never vanishing on ∂M) such that $\bar{g}:=\rho^2 g$ is a smooth Riemannian metric on $\bar{M}$. We will denote by $\hat{g}$ the induced metric on ∂M. Now if $|d\rho|_{\bar{g}}=1$ on ∂M, it is well known (see [Ma1] for instance) that g has asymptotically sectional curvature -1 near its boundary at infinity, in that case we say that (M, g) is *asymptotically hyperbolic*. If we assume more than (M, g) is Einstein, its forces it to satisfy

$$Ric(g)=-(n-1)\,g.$$

For any metric g on M, we denote by ∇ the associated Levi–Civita connexion, $Sect(g)$, $Riem(g)$, $Ric(g)$, $Scal(g)$ respectively the sectional, Riemann–Christoffel, Ricci, Scalar curvature of g. We let div denote the divergence $-tr\nabla$, its formal adjoint div^*, defined by

$$div^*\alpha(X, Y)=\tfrac{1}{2}\,(\nabla_X\alpha(Y)+\nabla_Y\alpha(X)),$$

and the Laplacian $\Delta=-tr\nabla\nabla$.

We denote by $\mathscr{T}_p^q$ the set of rank p covariant and rank q contravariant tensors. When $p=2$ and $q=0$, we denote by $\mathscr{S}_2$ the subset of symmetric tensors which split in $\mathscr{G}\oplus\mathscr{S}_{20}$ where $\mathscr{G}$ is the set of g-conformal tensors and

$\mathscr{S}_{20}$ the set of trace-free tensors (relatively to g). We observe the summation convention, and we use g_{ij} and its inverse g^{ij} to lower or raise indices.

Now we describe the Hölder spaces we work with (cf. [L1] and [GL]). Let $k \in \mathbb{N}$, $\alpha \in]0, 1[$, and $u \in C^k_{loc}(M)$. If M is a smooth open subset of $\mathbb{R}^n$, define

$$\|u\|^{(s)}_{k,0} = \sum_{|\gamma| \leqslant k} \sup_{x \in M} [d_x^{-s+|\gamma|} |\partial^\gamma u(x)|],$$

$$\|u\|^{(s)}_{k,\alpha} = \|u\|^{(s)}_{k,0} + \sum_{|\gamma| = k} \sup_{\substack{x,\, y \in B \\ x \neq y}} \min(d_x^{-s+k+\alpha}, d_y^{-s+k+\alpha}) \frac{|\partial^\gamma u(x) - \partial^\gamma u(y)|}{|x-y|^\alpha},$$

where d_x is the Euclidean distance from x to ∂M. We denote by $\Lambda^s_{k,\alpha}(M)$ the Banach set of the C^k_{loc} function on M for which $\|u\|^{(s)}_{k,\alpha}$ is finite. For tensors, we define by $\Lambda^s_{k,\alpha}(M, \mathscr{T}_p)$ the space of covariant rank p tensors on M the components of which in Euclidean coordinates are in $\Lambda^s_{k,\alpha}(M)$. For the general case, the same norms can be defined using a covering by background coordinates (smooth coordinate chart for an open subset of $\bar{M}$) and a subordinate partition of unity in the usual way. Note that these spaces have a lot of "good" properties [GL, Proposition 3.3].

Throughout this article we work on an asymptotically hyperbolic Einstein manifold (AHEM) denoted (M, g_0) and we are interested in metrics g which are close to g_0 in the spaces $\Lambda^s_{k,\alpha}(M, \mathscr{S}_2)$. The basic example of an AHEM is the real hyperbolic space (B, g_0) where B is the open unit ball of $\mathbb{R}^n$ and $g_0 = \rho^{-2}\bar{g}_0$ where $\bar{g}_0$ is the standard Euclidean metric and $\rho(x) = \frac{1}{2}(1 - |x|^2_{\bar{g}_0})$.

3. INVERTIBILITY OF THE LAPLACIAN

In this section, we study the invertibility of the operator $\Delta + \mathscr{K}$ acting on tensors, where Δ is the covariant Laplacian on (M, g_0) and $\mathscr{K}$ is a term of order zero. The main result is Theorem 3.1 which gives an isomorphism between suitable weighted Hölder spaces $\Lambda^s_{k,\alpha}$ introduced in Section 2.

THEOREM 3.1. *Let (M, g_0) be an AHEM with $\mathscr{Y}[\hat{g}_0] \geqslant 0$. Let $\alpha \in]0, 1[$, $k \in \mathbb{N}$, and let $\mathscr{K} \in \Lambda^0_{k,\alpha}(M, End(\mathscr{T}_p))$ be a self-adjoint endomorphism field. Define*

$$K := \inf g_0(\mathscr{K} u, u),$$

where the infimum is on $\{x \in M, u \in \mathscr{T}_{px}, g_0(u, u) = 1\}$. If

$$K > -\frac{(n-1)^2}{4},$$

define $s^{\pm} = s^{\pm}(\mathscr{K}) := \frac{n-1}{2} \pm \sqrt{(n-1)^2/4 + K}$. *Then the operator* $\Delta + \mathscr{K}$ *is an isomorphism from* $\Lambda^{s-p}_{k+2,\alpha}(M, \mathscr{T}_p)$ *to* $\Lambda^{s-p}_{k,\alpha}(M, \mathscr{T}_p)$ *for all* $s \in]\, s^-, s^+[$, $s \geqslant 0$.

Proof. From [GL, Propositions 3.7 and 3.8], for all $s \in]\, s^-, s^+[$, $s \geqslant 0$, it suffices to construct a function φ satisfying on M,[2]

$$\varphi \in C^2(M) \cap \Lambda^s_1(M), \qquad \rho^{-s}\varphi > C > 0 \qquad \text{and} \qquad (\Delta + K)\,\varphi > \delta\varphi,$$

for some constant C and δ. Now in [L1, Eq. (4.4)], Lee constructed a function

$$\varphi = u^{-s}, \qquad u = \rho^{-1} + v,$$

where $v \in \Lambda^0_{2,\alpha}(M)$, $u > 0$ on M, and $\Delta\varphi \geqslant s(n-1-s)\,\varphi$, if $s \geqslant 0$. As u is in $\Lambda^{-1}_{2,\alpha}(M)$ and $\rho u = 1 + \rho v$ is strictly positive on M and goes to 1 near ∂M, there exist $\varepsilon > 0$ such that $\rho u > \varepsilon$; by [GL, Proposition 3.3], we have that $\varphi = u^{-s}$ is in $\Lambda^s_{2,\alpha}(M)$. To complete the proof, it suffices to remark that there exists a constant C such that $\rho^{-s}\varphi = (\rho u)^{-s} > C > 0$ on M. ∎

Remark 3.2. This isomorphism is based on the construction of a function φ equivalent to ρ^s which satisfies $\Delta\varphi \geqslant s(n-1-s)\,\varphi$, in order to obtain the same interval of weight as for the hyperbolic space [GL]. In general such a function does not exist because its existence forces the first eigenvalue of the Laplacian on function to be bigger than $(n-1)^2/4$, while this is not the case for a large class of asymptotically hyperbolic manifolds [LP].

When $p = 2$, $\mathscr{T}_2$ splits into symmetric tensors $\mathscr{S}_2$ and antisymmetric tensors, and $\mathscr{S}_2$ splits further into g_0-conformal tensors $\mathscr{G}_0$ and trace-free tensors $\mathscr{S}_{20}$, and the Laplacian preserves these splittings. We then have:

COROLLARY 3.3. *Under conditions of Theorem* 3.1, *assume* $\mathscr{K} \in \Lambda^0_{k,\alpha}(M, End(\mathscr{S}_2))$ *is a self-adjoint endomorphism field. Then the operator* $\Delta + \mathscr{K}$ *is an isomorphism from* $\Lambda^{s-2}_{k+2,\alpha}(M, \mathscr{S}_{20})$ *to* $\Lambda^{s-2}_{k,\alpha}(M, \mathscr{S}_{20})$ *and from* $\Lambda^{s-2}_{k+2,\alpha}(M, \mathscr{G}_0)$ *to* $\Lambda^{s-2}_{k,\alpha}(M, \mathscr{G}_0)$ *for all* $s \in]\, s^-, s^+[$, $s \geqslant 0$.

Remark 3.4. (1) The condition $s \geqslant 0$ is perhaps not necessary as for the real hyperbolic space [GL], but it is sufficient for our purposes (see Remark 5.3).

(2) This result is not sharp on $\mathscr{S}_{20}$ because it uses a lower bound for the first eigenvalue of Δ on trace-free symmetric 2-tensor by that of

[2] Conditions in [GL, Proposition 3.8] are stronger than these but the proof goes through with these ones

function $(n-1)^2/4$. In [D2] I showed that this eigenvalue is in fact $(n-1)^2/4+2$ for the real hyperbolic space. Thus, the interval of weights might be somewhat bigger, but once again, we will see that it does not change the condition on the dimension for our purposes.

(3) If $K>0$, Lee [L1] has a better result on $\mathscr{G}_0$ (or equivalently on functions) without the assumptions $s \geqslant 0$ and $\mathscr{Y}[\hat{g}_0] \geqslant 0$.

4. RICCI CURVATURE

We show here that the Ricci equation

$$Ric(g) = R,$$

for a given symmetric 2-tensor R on M, is locally uniquely solvable in the neighborhood of the fixed AHEM metric g_0.

The linearized Ricci operator is [DT]

$$DRic(g)\, h = \tfrac{1}{2}\, \triangle_L h - div^*\, div\, grav\, h,$$

where $grav\, h := h - \frac{1}{2}\, tr(h)\, g$ and

$$\triangle_L = \triangle + \theta$$

is the Lichnerowicz Laplacian; recall

$$(\theta h)_{ij} = Ric(g)_{ik} h_j^k + Ric(g)_{jk} h_i^k - 2\, Sect(g)_{isjt}\, h^{st}.$$

It is well known that the Ricci operator is not elliptic because of the invariance by diffeomorphism and then satisfies the Bianchi identity

$$Bian(g, Ric(g)) = 0,$$

where $Bian(g, R) = -div\, grav\, R$. The linearized Bianchi operator in the first variable is

$$D_g\, Bian(g, R)\, h = \tilde{R}\, div\, grav\, h - T(g, R)\, h,$$

where $\tilde{R}_j^i := R_j^i$ and

$$[T(g, R)\, h]_j = T(g, R)_j^{kl}\, h_{kl} = \tfrac{1}{2}\, (\nabla^k R_j^l + \nabla^l R_j^k - \nabla_j R^{kl})\, h_{kl}.$$

In order to have a simple form for the linearization of the operator Q defined before, we will assume $T(g, R) = 0$. Note that in [DT] $T(g, R)$ was not written in that form and it is important for that fact: If R is a nondegenerate symmetric 2-tensor on M then we can see it as a semi-Riemannian metric with Christoffel's symbols Γ'. Denote by Γ Christoffel's symbols of g; then

$$T(g,R)_j^{kl}=R_{jp}\,g^{ki}g^{lq}(\Gamma'-\Gamma)_{iq}^p=\tilde{R}_j^s g_{ps}\,g^{ki}g^{lq}(\Gamma'-\Gamma)_{iq}^p.$$

In that situation, $T(g,R)=0$ is equivalent to the fact that g and R have the same Levi–Civita connexion.

LEMMA 4.1. *If g_0 is an Einstein metric then $T(g_0, Ric(g_0))=0$.*

Remark 4.2. As discussed above the condition to be Einstein is not necessary here. For example the Ricci metric ($\nabla Ric(g)=0$) satisfies $T(g, Ric(g))=0$.

DeTurck [DT] observed that solving the Ricci equation is equivalent to solving the system:

$$Q(g,R):=Ric(g)-R+div^*\,\tilde{R}^{-1}\,Bian(g,R)=0$$
$$Bian(g,R)=0.$$

The first equation is elliptic, while the second is elliptic underdetermined.

PROPOSITION 4.3. *Let (M, g_0) be an AHEM of dimension $n\geqslant 10$ with Yamabe invariant $\mathscr{Y}[\hat{g}_0]\geqslant 0$ and with sectional curvature less than a constant $c\geqslant -1$ which satisfies:*

$$K:=-4(n-1)-2(n-2)\,c>-\frac{(n-1)^2}{4}.$$

Let $k\in\mathbb{N}$, $k\geqslant 2$, $\alpha\in\,]0,1[$, and $s^{\pm}:=\frac{n-1}{2}\pm\sqrt{(n-1)^2/4+K}$. Then for all $s\in\,]s^-, s^+[$ and for any symmetric covariant two tensor close enough to $Ric(g_0)$ in $\Lambda_{k,\alpha}^{s-2}(M,\mathscr{S}_2)$, there exists a unique metric close to g_0 in the same space which solves

$$Q(g,R)=0.$$

Proof. The linearization of Q in the first variable at $(g_0, Ric(g_0))$ is

$$D_gQ(g_0, Ric(g_0))\,h=\tfrac{1}{2}\,\Delta_{g_0\,L}h=\tfrac{1}{2}\,(\Delta_{g_0}+\theta(g_0))\,h.$$

Decompose $h=ug_0+h_0$ across the splitting $\mathscr{S}_2=\mathscr{G}_0\oplus\mathscr{S}_{20}$; then

$$\theta(g_0)\,h=-2(n-1)\,h_0-2\,Sect(g_0)\,h_0.$$

Now by the Fujitani lemma (see [Be, Lemma 12.71, p. 356]), as the sectional curvature is less than c we have

$$g_0(\theta h_0, h_0) \geqslant K g_0(h_0, h_0).$$

Corollary 3.3 and the implicit function theorem complete the proof. ∎

PROPOSITION 4.4. *Let (M, g_0) be an AHEM of dimension $n \geqslant 6$ with Yamabe invariant $\mathscr{Y}[\hat{g}_0] \geqslant 0$. Let $k \in \mathbb{N}$, $k \geqslant 3$, $\alpha \in]0, 1[$, and $t^{\pm} := \frac{n-1}{2} \pm \sqrt{(n-1)^2/4 - (n-1)}$. Then for all $s \in]t^-, t^+[$ and for any symmetric covariant two tensor close enough to $Ric(g_0)$ in $\Lambda^{s-2}_{k,\alpha}(M, \mathscr{S}_2)$, if g is a metric close enough to g_0 in the same space and solve $Q(g, R) = 0$, then g is a solution of the Ricci equation*

$$Ric(g) = R.$$

Proof. Apply the Bianchi operator to the equation $Q(g, R) = 0$:

$$0 = Bian(g, Q(g, R)) = -Bian(g, R) + Bian(g, div^* \tilde{R}^{-1} Bian(g, R)). \quad (4.1)$$

For all 1-forms ω, we have (see [D1, Lemma 2] for example)

$$Bian(g, div^* \omega)_m = \tfrac{1}{2}(-\Delta\omega_m + Ric(g)_{km}\, \omega^k).$$

Let us consider the operator on 1-forms

$$L_{g,R}\, \omega_m := \Delta(\tilde{R}^{-1}\omega)_m - Ric(g)_{km}\, (R^{-1})^{kl}\, \omega_l + 2\omega_m.$$

For $g = g_0$ and $R = Ric(g_0)$, Theorem 3.1 asserts that

$$L_{g_0, Ric(g_0)} = -\frac{1}{n-1}[\Delta_{g_0} - (n-1)]$$

is an isomorphism from $\Lambda^{s-1}_{k-1,\alpha}(M, \mathscr{T}_1)$ to $\Lambda^{s-1}_{k-3,\alpha}(M, \mathscr{T}_1)$ for all $s \in]t^-, t^+[$. Then there exists a constant $C > 0$ such that

$$\|L_{g_0}\omega\|^{(s-1)}_{k-3,\alpha} > C\, \|\omega\|^{(s-1)}_{k-1,\alpha}.$$

Now it is easy to see (cf. [D]) that if g is close enough to g_0 in $\Lambda^{-2}_{k,\alpha}(M, \mathscr{S}_2)$ and R close enough to $Ric(g_0)$ in the same space, we have

$$\|L_{g,R} - L_{g_0, Ric(g_0)}\| < \tfrac{1}{2} C,$$

where the norm is the usual norm for a linear map. Let $\omega := Bian(g, R)$. From Eq. (4.1), we have $L_{g,R}\,\omega = 0$. Thus

$$C\,\|\omega\|^{(s-1)}_{k-1,\alpha} \leqslant \|L_{g_0, Ric(g_0)}\omega\|^{(s-1)}_{k-3,\alpha} = \|(L_{g,R} - L_{g_0, Ric(g_0)})\,\omega\|^{(s-1)}_{k-3,\alpha} \leqslant \tfrac{1}{2}\,C\,\|\omega\|^{(s-1)}_{k-1,\alpha};$$

hence $\omega = 0$. ∎

From Propositions 4.3 and 4.4 we deduce

THEOREM 4.5. *Let (M, g_0) be an AHEM of dimension $n \geqslant 10$ with Yamabe invariant $\mathscr{Y}[\hat{g}_0] \geqslant 0$ and with sectional curvature less than $c \geqslant -1$ and which satisfies:*

$$K := -4(n-1) - 2(n-2)\,c > -\frac{(n-1)^2}{4}.$$

Let $k \in \mathbb{N}$, $k \geqslant 2$, $\alpha \in\,]0, 1[$, and $s^{\pm} := \frac{n-1}{2} \pm \sqrt{(n-1)^2/4 + K}$. Then for all $s \in\,]s^-, s^+[$, any symmetric covariant two tensor close enough to $Ric(g_0)$ in $\Lambda^{s-2}_{k,\alpha}(M, \mathscr{S}_2)$ is the Ricci curvature of a unique metric close to g_0 in the same space.

Remark 4.6. As discussed in Section 3, the interval of weight $]s^-, s^+[$ is not sharp and might be a little bigger as in [DH], this is due to the condition $2n < (n-1)^2/4$ and the best condition we could hope for is $2n < (n-1)^2/4 + 2$ [D2]. Note that in that case it *does not change* the condition on the dimension.

The solution we obtain is in $\Lambda^{s-2}_{k,\alpha}$ and we do not know if it is $\Lambda^{s-2}_{k+2,\alpha}$ as we could hope; this is due to the addition of the Bianchi term in the Ricci equation. To remedy this problem we define the Frechet spaces Λ^{s-2}_{∞} with the family of norm $\|.\|^{(s)}_{k,\alpha}$, $k \in \mathbb{N}$ (α fixed in $]0, 1[$). As in [D1] and without using the Nash–Moser inverse function theorem, we obtain:

THEOREM 4.7. *Under conditions of Theorem 4.5, for all $s \in\,]s^-, s^+[$ we have*

(1) *Any symmetric covariant two tensor close enough to $Ric(g_0)$ in $\Lambda^{s-2}_{\infty}(M, \mathscr{S}_2)$ is the Ricci curvature of a unique metric close to g_0 in the same space.*

(2) *The image of the Riemann Christoffel operator*

$$h \to Riem(g_0 + h) - Riem(g_0)$$

is a submanifold in a neighborhood of zero in Λ^{s-2}_{∞}.

5. EINSTEIN CURVATURE

We study here the Einstein equation with cosmological constant

$$Ein(g) := Ric(g) - \tfrac{1}{2}\, Scal(g)\, g + \Lambda g = E,$$

where E is any symmetric covariant 2-tensor close to $Ein(g_0)$, g_0 is an AHEM, and Λ is a constant.

Remark first that

$$Ein(g) = grav(Ric(g)) + \Lambda g.$$

As $grav^{-1} T = T - \frac{1}{n-2}\, tr(T)\, g$, solving the Einstein equation is equivalent to solving

$$Ric(g) = grav^{-1}(E) + \frac{2}{n-2}\Lambda g.$$

This last equation is the same as the Ricci one modulo a term of order zero. We will reproduce step by step Section 4. If $Ein(g) = E$, the Bianchi identity gives

$$div\, E = 0.$$

Note that $div\, E = div\; grav\; E - \frac{1}{2}\nabla tr(E)$, hence

$$D_g(div\, E)(g)\, h_i = -(\tilde{E}\; div\; grav\; h)_i + \tfrac{1}{2}\,(\nabla_i h^{st})\, E_{st} + U(g, E)_i^{kl}\, h_{kl},$$

where $U(g, E)_i^{kl} = g^{ks} g^{lt} \nabla_s E_{ti}$. We have

Lemma 5.1. *If g is Einstein then $U(g, Ein(g)) = 0$.*

We are going to solve the system equivalent to the Einstein equation:

$$A(g, E) := Ric(g) - grav^{-1}\, E - \frac{2}{n-2}\Lambda g - div^*\, \tilde{E}^{-1}\, div\, E = 0$$

$$div\, E = 0$$

Proposition 5.2. *Let (M, g_0) be an AHEM of dimension $n \geqslant 3$ with Yamabe invariant $\mathcal{Y}[\hat{g}_0] \geqslant 0$ and with sectional curvature less than $c \geqslant -1$. Assume*

$$K := \min\left(-\frac{2}{n-2}\Lambda,\ (n-1)(n-3) - 2(n-2)\, c - \frac{4}{n-2}\Lambda\right) > -\frac{(n-1)^2}{4}$$

and $\Lambda \neq -\frac{(n-1)(n-2)}{2}$. *Let* $k \in \mathbb{N}$, $k \geqslant 2$, $\alpha \in]0,1[$, *and* $s^{\pm} := \frac{n-1}{2} \pm \sqrt{(n-1)^2/4+K}$. *Then for all* $s \in]s^-, s^+[$, $s \geqslant 0$, *and any symmetric covariant two tensor close enough to* $Ein(g_0)$ *in* $\Lambda^{s-2}_{k,\alpha}(M, \mathcal{S}_2)$, *there exists a unique metric close to* g_0 *in the same space which solves*

$$A(g, E) = 0.$$

Remark 5.3. The assumption $s \geqslant 0$ is not only used for Corollary 3.3, but we need it to be sure that a symmetric tensor near g_0 in $\Lambda^{s-2}_{k,\alpha}(M, \mathcal{S}_2)$ stays positive definite, that E stays invertible near $Ein(g_0)$, and that $Ein(g_0+h) - Ein(g_0)$ is in $\Lambda^{s-2}_{k,\alpha}$ for $h \in \Lambda^{s-2}_{k+2,\alpha}$ close to zero.

Proof. The fact that $\Lambda \neq -\frac{(n-1)(n-2)}{2}$ guarantees that

$$Ein(g_0) = \left[\frac{(n-1)(n-2)}{2} + \Lambda\right] g_0$$

is invertible and then E is invertible if it is close enough to $Ein(g_0)$. The linearisation of the operator A in the first variable at $(g_0, Ein(g_0))$ is

$$D_g A(g_0, Ein(g_0))\, h = \frac{1}{2} \Delta_{g_0 L} h + \frac{n-1}{2}(nh - tr(h)\, g_0) - \frac{2}{n-2} \Lambda h - \frac{1}{2} \nabla\nabla(tr(h)).$$

We write $h = ug_0 + h_0$ using the splitting $\mathcal{S}_2 = \mathcal{G}_0 \oplus \mathcal{S}_{20}$; thus we obtain

$$2D_g A(g_0, Ein(g_0)) = \underbrace{2\left(\Delta_{g_0} - \frac{2}{n-2}\Lambda\right) ug_0}_{\in \mathcal{G}_0}$$

$$+ \underbrace{\left[\Delta_{g_0} + (n-1)(n-2) - \frac{4}{n-2}\Lambda - 2\, Sect(g_0)\right] h_0 - n\nabla_{g_0}\nabla_{g_0} u - \Delta_{g_0} u g_0}_{\in \mathcal{S}_{20}}.$$

From the Fujitani lemma [Be, Lemma 12.71] and Corollary 3.3, this operator is an isomorphism from $\Lambda^{s-2}_{k,\alpha}(M, \mathcal{S}_2)$ to $\Lambda^{s-2}_{k-2,\alpha}(M, \mathcal{S}_2)$ if $s \geqslant 0$ is in $]s^-, s^+[$. The implicit function theorem completes the proof. ∎

PROPOSITION 5.4. *Let* (M, g_0) *be an AHEM of dimension* $n \geqslant 3$ *with Yamabe invariant* $\mathcal{Y}[\hat{g}_0] \geqslant 0$. *Let* $k \in \mathbb{N}$, $k \geqslant 3$, $\alpha \in]0,1[$, *and* $s^{\pm} := \frac{n-1}{2} \pm \sqrt{(n-1)^2/4 + (n-1)^2}$. *Then for all* $s \in]s^-, s^+[$, $s \geqslant 0$, *and any symmetric covariant two tensors close enough to* $Ein(g_0)$ *in* $\Lambda^{s-2}_{k,\alpha}(M, \mathcal{S}_2)$, *if* g *is a metric close enough to* g_0 *in the same space and solves* $A(g, E) = 0$, *then* g *is a solution of the Einstein equation*

$$Ein(g) = E.$$

Proof. Apply the Bianchi operator to the equation $A(g, E)=0$:

$$0 = Bian(g, A(g, E)) = -div\ E + Bian(g, div^*\ \tilde{E}\ div\ E).$$

Let us consider the operator on 1-forms

$$L_{g,E} w_m := \triangle(\tilde{E}^{-1}\omega)_m - Ric(g)_{ms}\ (E^{-1})^{sl}\ \omega_l + 2\omega_m.$$

For $g = g_0$ and $E = Ein(g_0)$, we have $L_{g_0, Ein(g_0)} = \frac{2}{(n-1)(n-2)} [\triangle_{g_0} + (n-1)^2]$ and the proof is the same as for Proposition 4.4. ∎

From Propositions 5.2 and 5.4, we obtain

THEOREM 5.5. *Let (M, g_0) be an AHEM of dimension $n \geqslant 3$ with Yamabe invariant $\mathscr{Y}[\hat{g}_0] \geqslant 0$ and with sectional curvature less than c. Assume*

$$K := \min\left(-\frac{2}{n-2}\Lambda,\ (n-1)(n-3) - 2(n-2)\,c - \frac{4}{n-2}\Lambda,\ (n-1)^2\right)$$
$$> -\frac{(n-1)^2}{4}$$

and $\Lambda \neq -\frac{(n-1)(n-2)}{2}$. Let $k \in \mathbb{N}$, $k \geqslant 3$, $\alpha \in]0, 1[$, and $s^{\pm} := \frac{n-1}{2} \pm \sqrt{(n-1)^2/4 + K}$. Then for all $s \in]s^-, s^+[$, $s \geqslant 0$, and any symmetric covariant two tensor close enough to $Ein(g_0)$ in $\Lambda^{s-2}_{k,\alpha}(M, \mathscr{S}_2)$, there exist a unique metric close to g_0 in the same space which solves the Einstein equation

$$Ein(g) = E.$$

Remark 5.6. (1) The case $\Lambda \geqslant 0$ (then $s > 0$) forces the metric to be asymptotically g_0 at infinity but not when $\Lambda < 0$.

(2) When $K = -\frac{2}{n-2}\Lambda$, the interval of weights is sharp because the condition comes from the action of the Laplacian function.

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